

UNITARY REPRESENTATIONS OF UNIMODULAR LIE GROUPS IN BERGMAN SPACES

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ABSTRACT. For G an arbitrary unimodular Lie group, we construct strongly continuous unitary representations of G in the Bergman space of a naturally constructed strongly pseudoconvex neighborhood of the complexification of the manifold underlying G .

1. INTRODUCTION

Let G be a Lie group of real dimension n , acting freely by real-analytic transformations on a m -dimensional, C^ω manifold X . In [HHK] it is shown that any such G action can be extended to a neighborhood of X in its complexification $X^\mathbb{C} \supset X$, and in this neighborhood, the extended transformations can be chosen to be biholomorphisms. In the same work, the authors also construct a G -invariant strongly plurisubharmonic function φ which vanishes on X , thus by setting

$$(1) \quad M_\epsilon = \{\varphi < \epsilon\} \subset X^\mathbb{C}$$

for $\epsilon > 0$ sufficiently small, one obtains a strongly pseudoconvex G -manifold M_ϵ , topologically equal to the Cartesian product of X with a real n -dimensional ball, on which G acts freely by holomorphic transformations. The spaces M_ϵ are called *gauged G -complexifications of X* in [HHK] and elsewhere are frequently called *Grauert tubes*. By construction, the M_ϵ are Stein manifolds (see also [Gr]) and so possess a rich collection of holomorphic functions $\mathcal{O}(M_\epsilon)$ which is invariant under the induced group action.

Choosing an invariant measure on M_ϵ we may analyze the subspace of $\mathcal{O}(M_\epsilon)$ consisting of those members which are square-integrable. This is called the Bergman space of M_ϵ and we will denote it $L^2\mathcal{O}(M_\epsilon)$. The main goal of this paper is to show that, when G is unimodular, the

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Bergman space of M_ϵ , is large enough to furnish non-trivial unitary representations of G . The techniques we use are adaptations of the method of solution of the Levi problem as [K1, FK] in the compact case and which were developed and adapted for noncompact G -manifolds in [GHS, P1, P2].

Our main result is

Theorem 1.1. *If G is a unimodular Lie group of dimension n , then there exist a complex manifold M , which is topologically the Cartesian product of G with an n -ball, and a strongly continuous unitary representation \mathcal{R} of G in the Bergman space of M such that $\ker \mathcal{R}$ is a compact subgroup of G .*

Corollary 1.2. *If G has no compact subgroups, then the unitary representations constructed here are faithful.*

Remark 1.3. In [GHS], a G -manifold with a nonunimodular structure group is constructed which is Stein but has $L^2\mathcal{O} = \{0\}$.

Our methods are mainly geometric in the present article as we have previously provided analytic sufficient conditions in [P2] for the existence of a nontrivial, and in fact large, Bergman space on some tubes. Let us describe these methods here briefly.

Mimicking the solution of the Levi problem in [K1, GHS], L^2 -holomorphic functions were constructed in [P2] on G -manifolds M with compact quotient \bar{M}/G if the following main assumptions are fulfilled. First, we assume that the group G is unimodular. Second, the complex manifold M is assumed strongly pseudoconvex. Third, roughly speaking, some negative power of a Levi polynomial is required to have the property that convolutions by the group action not smooth its singularities at the boundary of M . This last property was dubbed *amenability* in [P2] and much of the present article will be concerned with demonstrating that it holds for some tubes.

2. PRELIMINARIES

2.1. Local properties of Levi's polynomial. The main technical material in this paper will concern local details of Levi's polynomial. We will thus go through a fairly thorough description of this function here. Let M be a complex manifold with nonempty smooth boundary bM , $\bar{M} = M \cup bM$, so that M is the interior of \bar{M} , and $\dim_{\mathbb{C}}(M) = n$. We will also assume for simplicity that \bar{M} is a closed subset in $\widetilde{\bar{M}}$, a complex neighborhood of \bar{M} so that the complex structure on $\widetilde{\bar{M}}$ extends that of M , and every point of \bar{M} is an interior point of $\widetilde{\bar{M}}$.

Let us choose a smooth function $\rho : \widetilde{M} \rightarrow \mathbb{R}$ so that

$$M = \{z \mid \rho(z) < 0\}, \quad bM = \{z \mid \rho(z) = 0\},$$

and for all $x \in bM$, we have $d\rho(x) \neq 0$. For any $x \in bM$ define the *holomorphic tangent plane* to the boundary at x by

$$T_x^{\mathbb{C}}(bM) = \{w \in \mathbb{C}^n \mid \sum_{k=1}^n \frac{\partial \rho}{\partial z_k} \Big|_x w_k = 0\}.$$

For $x \in bM$, define the Levi form L_x by

$$L_x(w, \bar{w}) = \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \Big|_x w_j \bar{w}_k, \quad (w \in T_x^{\mathbb{C}}(bM)).$$

Then M is said to be *strongly pseudoconvex* if for every $x \in bM$, the form L_x is positive definite. Though L_x depends on ρ , its essential features are preserved by biholomorphisms.

Since ρ is real-valued, the Taylor expansion at x of ρ is

$$(2) \quad \rho(z) = \rho(x) + 2\Re f(z, x) + L_x(z - x, \bar{z} - \bar{x}) + \mathcal{O}(|z - x|^3), \quad (z \in \mathbb{C}^n)$$

with the *Levi polynomial* f defined by

$$(3) \quad f(z, x) = \sum_{k=1}^n \frac{\partial \rho}{\partial z_k} \Big|_x (z_k - x_k) + \frac{1}{2} \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \Big|_x (z_j - x_j)(\bar{z}_k - \bar{x}_k).$$

As discussed in the introduction, we need to know when convolutions of the singular functions gotten by taking the Levi polynomial to negative powers are not smooth in the boundary. We start with an analysis of f itself. The content of the next two paragraphs is well-known.

In our Grauert tubes, the group invariance and the compactness of the quotient guarantee that without loss of generality (replacing ρ by $e^{\lambda\rho} - 1$ with sufficiently large $\lambda > 0$) we may choose a defining function of M so that the Levi form $L_x(w, \bar{w})$ is positive for all nonzero $w \in \mathbb{C}^n$ (and not only for $w \in T_x^{\mathbb{C}}(bM)$) and at all points $x \in bM$.

The complex quadric hypersurface $S_x = \{z \mid f(z, x) = 0\}$ has $T_x^{\mathbb{C}}(bM)$ as its tangent plane at x . The strong pseudoconvexity property implies that $\rho(z) > 0$ if $f(z, x) = 0$ and $z \neq x$ is close to x . This means that near x the intersection of S_x with bM contains only x . Since $\rho < 0$ in M , (2) implies that $\Re f(z, x) < 0$ if $x \in bM$ and $z \in M$ is sufficiently close to x . It follows that we can choose a branch of $\log f(z, x)$ so that $z \mapsto \log f(z, x)$ is a holomorphic function in $z \in M \cap U_x$, where U_x is a sufficiently small neighborhood of x in

M . Consequently all powers of f are also well-defined and holomorphic in U_x . For $\tau < 0$, the functions $f^\tau : U_x \rightarrow \mathbb{C}$ are holomorphic in a neighborhood of x and blow up only at x .

Lemma 2.1. *Choose local coordinates for which $x \cong 0$ and let*

$$a = \left. \frac{\partial \rho}{\partial z_k} \right|_0 \quad \text{and} \quad b = \frac{1}{2} \sum_{j,k=1}^n \left. \frac{\partial^2 \rho}{\partial z_j \partial z_k} \right|_0$$

so that $f(z, 0) = a \cdot z + bz \cdot z$. It follows that for z sufficiently near zero in \bar{M} there are constants $C, D > 0$ so that

$$C|z|^2 \leq |a \cdot z + bz \cdot z| \leq D|z|.$$

Proof. This is true because

$$\begin{aligned} 2|a \cdot z + bz \cdot z| &\geq -2\Re(a \cdot z + bz \cdot z) \\ &\geq \rho(z) - 2\Re(a \cdot z + bz \cdot z) = L_0(z, \bar{z}) + \mathcal{O}(|z|^3) \end{aligned}$$

and the Levi form has a smallest eigenvalue $\lambda > 0$, so $L_0(z, \bar{z}) > \lambda|z|^2$. The other estimate is obvious. \square

The estimate above yields the following immediately, [P2, Lemma 4.2].

Lemma 2.2. *Let M be a complex manifold, $n = \dim_{\mathbb{C}} M$, and $\chi \in C_c^\infty(\bar{M})$ with small support near zero. Then $\chi f^{-\tau} \in L^p(M)$ whenever $\tau \in (0, n/p)$.*

2.2. Grauert tubes. As we mentioned in the introduction, if X is a real-analytic manifold on which a Lie group G acts freely by real-analytic transformations, then there exists a complexification $X^{\mathbb{C}}$ so that $X \hookrightarrow X^{\mathbb{C}}$ is embedded as a totally real submanifold. It is shown in [HHK] that the action of G on X extends to a neighborhood V of X in $X^{\mathbb{C}}$ such that the extended transformations are biholomorphisms. Furthermore, there exists a real-valued function $\varphi \in C^\omega(V, \mathbb{R})$ with the following properties:

- (1) φ is constant along the orbits of G
- (2) $\varphi|_X \equiv 0$
- (3) $\varphi \geq 0$ is strongly plurisubharmonic near X .

Following [HHK], define $M_\epsilon = \{\varphi < \epsilon\} \subset V$. Note that G acts freely on M_ϵ by holomorphic transformations and M_ϵ is strongly pseudoconvex as its boundary is the level set of a strongly plurisubharmonic function.

In [HHK], it is also shown that there is an equivariant retraction $R(t, z)$, $t \in [0, 1]$, from a neighborhood of X in M_ϵ onto X ; in particular

the map $R(1, z)$ is a projection $\pi : M_\epsilon \rightarrow X$ commuting with the right G -action.

We will begin our analysis of M_ϵ by examining the Taylor series of φ , *cf.* VIII, Lemma 1, [HHK].

Lemma 2.3. *Let M_ϵ and φ be as before, and let $p \in M_\epsilon$. There exist local complex coordinates $z_j = x_j + iy_j$, $j = 1, \dots, m$, vanishing at p , such that $\varphi(z) = \sum_j y_j^2 + \mathcal{O}(|z|^3)$.*

Proof. Since $X \subset M_\epsilon$ is totally real, with a holomorphic change of coordinates, we may assume that $X = \{y_j = 0\}$. Since each point of X is a local minimum for φ , we have $\nabla\varphi|_X \equiv 0$, which implies $\partial^2\varphi/\partial x_k \partial y_j(0) = 0$. Since we also have $\partial^2\varphi/\partial x_k \partial x_j(0) = 0$, we obtain

$$\frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}(0) = \frac{\partial^2\varphi}{\partial y_j \partial y_k}(0).$$

Since φ is strongly plurisubharmonic, the form on the left-hand side is positive definite, it follows that the second-order Taylor expansion of φ about $0 \in \mathbb{C}^m$ is a positive definite, purely quadratic polynomial involving only the y_j . The claim is obtained by diagonalizing the real, symmetric form. \square

3. CONVOLUTIONS OF LEVI'S POLYNOMIALS

From now on we restrict our attention to the case in which M_ϵ is the gauged G -complexification of G acting on itself, *i.e.* $X = G$, $M_\epsilon \subset G^\mathbb{C}$. We have been unable to demonstrate that M_ϵ possesses the amenability property mentioned in the introduction (see Definition 3.7). As a result, we *thicken* it in the following way.

With z_0 a coordinate on $T = (S^1)^\mathbb{C} \cong \mathbb{C}/\mathbb{Z}$, and z' a coordinate in $M_\delta \subset V$ from before, define the function $\tilde{\varphi}(z_0, z') = (\Im z_0)^2 + \varphi(z')$ on $T \times M_\delta$ with $\delta > 0$ sufficiently small. As before, define

$$\tilde{M}_\epsilon = \{\tilde{\varphi} < \epsilon\}$$

for $0 < \epsilon < \delta$. By construction $\tilde{\varphi}$ is strictly plurisubharmonic, thus \tilde{M}_ϵ is strongly pseudoconvex for $\epsilon > 0$ sufficiently small. Extending the G action to $T \times M_\delta$ by triviality on the T factor, we see that G acts freely on \tilde{M}_ϵ by holomorphic transformations, and with compact quotient. Because the manifold \tilde{M}_ϵ admits a free G action,

$$(4) \quad \tilde{M}_\epsilon = G \times \mathbb{T}$$

where \mathbb{T} is a solid torus of $n + 2$ real dimensions. Indeed, at each point of the group, we have $\Re z_0 \in S^1$ free and $\{y = (\Im z_0, y_1, \dots, y_n) \mid (\Im z_0)^2 + \sum_1^n y_k^2 < \epsilon\} \cong B^{n+1}$ and we have denoted $\mathbb{T} = S^1 \times B^{n+1}$.

We compute the Levi polynomial at the basepoint $(z_0, z') = (i\epsilon, 0)$ of the boundary of \tilde{M}_ϵ in the local complex coordinates $z = (z_0, z_1, \dots, z_n)$.

Lemma 3.1. *The Levi polynomial induced by the defining function $\tilde{\varphi}(z)$ at the point $p = (i\epsilon, 0)$ is*

$$f(z) = 4\epsilon - 1 + 4iz_0(2\epsilon - 1) + 4z_0^2 + 4z_1^2 + \dots + 4z_n^2.$$

Proof. Follows directly from the form of φ . \square

In the following, we are going to denote by zt the action of $t \in G$ on $z \in \tilde{M}_\epsilon$.

Proposition 3.2. *In \tilde{M}_ϵ , consider the curve $(0, \epsilon] \ni s \mapsto (is, 0)$ to the basepoint $(i\epsilon, 0)$. There exist coordinates $t = (t_j)_1^n$ in a neighborhood of $e \in G$ such that, for z belonging to the curve, the Levi polynomial at the point zt takes the form*

$$f(zt) = \sigma + t_1^2 + \dots + t_n^2$$

and $\sigma = \sigma(s) \rightarrow 0$ as $s \rightarrow \epsilon$.

Proof. As $G = X$, the construction in Lemma 2.3 gives real coordinates for G near the identity $e \in G$. We compute $f(zt) = f(z_0, z't)$ along the path obtaining

$$f((is, e)t) = f(is, t) = 4\epsilon - 1 + 4is(2\epsilon - 1) + 4s^2 + 4t_1^2 + \dots + 4t_n^2$$

The $\sigma(s)$ from the claim is simply $4\epsilon - 1 + 4is(2\epsilon - 1) + 4s^2$. \square

Remark 3.3. What we will use of this proposition is simply the fact that the Levi polynomial on the orbit of this special curve takes the form of a constant plus a norm-like quantity on G .

On our tubes, thickened and unthickened, we have smooth, free, right actions of G with compact quotients. Choosing a biinvariant measure dt on G , and fixing smooth measures dq on B^n and dQ on \mathbb{T} , the tensor product measures on the tubes allow us to decompose $L^2(M_\epsilon)$ and $L^2(\tilde{M}_\epsilon)$ as follows

$$(5) \quad \begin{aligned} L^2(M_\epsilon, dt \otimes dq) &\cong L^2(G, dt) \otimes L^2(B^n, dq) \\ L^2(\tilde{M}_\epsilon, dt \otimes dQ) &\cong L^2(G, dt) \otimes L^2(\mathbb{T}, dQ), \end{aligned}$$

which also present the Hilbert spaces as free Hilbert G -modules, [GHS]. Later, dq and dQ will be chosen in a coherent way, but for now we think of them as being arbitrary smooth measures.

On these manifolds, the global right G -action and the Haar measure dt combine to define convolution operators in L^2 which we will write

$$(R_\Delta u)(z) = \int_G dt \Delta(t) u(zt),$$

for example, for $\Delta \in L^1(G)$, $u \in L^2$.

Let us begin our analysis of the asymptotics of convolutions of powers of f . For U_x a coordinate neighborhood at the boundary of \tilde{M}_ϵ choose a cut-off function $\chi \in C_c^\infty(U_x)$, so that $\chi = 1$ in a neighborhood of x .

Lemma 3.4. [P2, Rem. 4.3] *Let M be a strongly pseudoconvex G -manifold. Assume for simplicity that M is a trivial G -bundle and let f be a Levi polynomial on M . Then for all $x \in \bar{M}/G$, $\|\chi f^{-\tau}(\cdot, x)\|_{L^1(G)} < \infty$ as long as $2\tau < \dim_{\mathbb{R}} G$.*

We will also need the following fact regarding convolutions of powers of Levi's polynomial.

Lemma 3.5. *Let $n = \dim_{\mathbb{R}} G$, $\tau \in (0, n/2)$ and $\Delta \in L^2(G)$. Then $R_\Delta \chi f^{-\tau} \in L^2(M)$.*

Proof. Since $\text{meas}(M/G) < \infty$, Young's and Jensen's inequalities give

$$\begin{aligned}
 (6) \quad \|R_\Delta h\|_{L^2(M)}^2 &= \int_{M/G} dx \int_G dt \left| \int_G ds \Delta(s) h(ts, x) \right|^2 \\
 &\leq \|\Delta\|_{L^2(G)}^2 \int_{M/G} dx \|h(\cdot, x)\|_{L^1(G)}^2 \\
 &\lesssim \|\Delta\|_{L^2(G)}^2 \left| \int_{M/G} dx \|h(\cdot, x)\|_{L^1(G)} \right|^2 \\
 &= \|\Delta\|_{L^2(G)}^2 \|h\|_{L^1(M)}^2.
 \end{aligned}$$

Lemmata 2.2 and 3.4 provide that $h = \chi f^{-\tau} \in L^1(M)$, which gives the result. \square

Remark 3.6. Note that in the expression (6), with $h = \chi f^{-\tau}$, the G -integral is over a compact neighborhood of the identity. This neighborhood can be chosen as small as we like, choosing χ accordingly.

As we have mentioned in the introduction, in [P2] a sufficient condition is given guaranteeing that the Bergman space of any strongly pseudoconvex G -manifold with compact quotient be large in the sense of von Neumann's G -dimension. It is the following

Definition 3.7. Let $\xi : \bar{Y} \rightarrow M$ be a piecewise continuous section of $G \rightarrow M \xrightarrow{p} Y$ so that $\xi|_{p(\text{supp } \chi)}$ is continuous. The action of G on M is called *amenable* if there exist an $x \in bM$ and $\tau > 0$ so that if f is a Levi polynomial at x , then 1) $\chi f^{-\tau} \in L^2(M)$, 2) $\|\chi f^{-\tau}(\cdot, \xi)\|_{L^1(G)} < \infty$ for all $\xi \in \bar{Y}$, and 3) $R_\Delta \chi f^{-\tau} \notin C^\infty(\bar{M})$ for all nonzero $\Delta \in C^\infty(G)$.

Remark 3.8. The reader will note that the formulation of the definition seems to be dependent on the defining function.

Proposition 3.9. *The G action on the manifold \tilde{M}_ϵ is amenable.*

Proof. First note that if $\dim_{\mathbb{R}} G = n$, then $\dim_{\mathbb{C}} \tilde{M}_\epsilon = n + 1$, thus

- (1) $\chi f^{-\tau} \in L^2(M)$ if $\tau \in [0, \frac{n+1}{2})$
- (2) $\|\chi f^{-\tau}(\cdot, \xi)\|_{L^1(G)} < \infty$ for all $\xi \in X$ if $2\tau < \dim_{\mathbb{R}} G = n$.

by Lemmata 2.2 and 3.4, respectively. So let us choose $\tau < n/2$ and estimate the convolution along a path to the basepoint in $b\tilde{M}_\epsilon$, using Prop. 3.2:

$$\begin{aligned} (R_\Delta f^{-\tau})(is/2, 0) &\sim \int dt [4\epsilon - 1 - 2(2\epsilon - 1)s - s^2 + 4|t|^2]^{-\tau} \\ &\sim \int_0^\delta \frac{r^{n-1} dr}{[4\epsilon - 1 - 2(2\epsilon - 1)s - s^2 + 4r^2]^\tau} \\ &\sim \int_0^\delta \frac{r^{n-1} dr}{[\sigma + 4r^2]^\tau} \end{aligned}$$

where $\sigma \rightarrow 0$ as $s \rightarrow 1$, and where we have invoked the fact in Rem. 3.6. Away from the path of the singularity, the convolution is smooth, thus we have

$$\lim_{s \rightarrow 1} \frac{\partial^k}{\partial s^k} (R_\Delta f^{-\tau})(is/2, 0) \longrightarrow \infty$$

for $\tau > 0$, $\tau + k > n$. Taking $\tau = 1$ suffices. \square

Remark 3.10. Estimates of convolutions in substantially greater generality suggest that the amenability property is satisfied whenever the group action avoids the “bad” direction. For example, in the language of [FS], subgroups of the Heisenberg group of the form $\{(z, 0) \mid z \in \mathbb{C}^n\}$ lead to amenable actions while those containing $\{(0, t) \mid t \in \mathbb{R}\}$ do not. This tempts us to conjecture that in the gauged G -complexification M of a G -manifold X for which $\dim_{\mathbb{R}} X > \dim_{\mathbb{R}} G$, the extended G -action on M is amenable.

Theorem 3.11. $\dim_G L^2 \mathcal{O}(\tilde{M}_\epsilon) = \infty$

Proof. \tilde{M}_ϵ satisfies the requirements of Thm. 5.2 in [P2]. \square

Remark 3.12. The positivity of the G -dimension is sufficient to obtain the infinite-dimensionality of the Bergman space of \tilde{M}_ϵ over the complex numbers.

4. BERGMAN REPRESENTATION SPACES

4.1. Preparatory geometric considerations. Any G -invariant Riemannian metric on \tilde{M} is complete in the following sense. For any point

$x_0 \in \bar{M}$ and for any $R > 0$ the ball $B(t, R) = \{s \in \bar{M} : \text{dist}(t, s) < R\}$ of the corresponding geodesic metric is relatively compact in \bar{M} , [GHS].

We will need the following topological lemma. For $K \subset G$ and $t \in G$, denote by Kt the set $\{kt : k \in K\} \subset G$.

Lemma 4.1. *Let G be a non-compact Lie group, $K \subset G$ a compact subset containing the identity, and $L \subset G$ an unbounded sequence. It follows that there exists a $t \in L$ such that $K \cap Kt = \emptyset$.*

Proof. Choose a right-invariant Riemannian metric d on the group G . By the observation above, G 's closed and d -bounded subsets are compact. For K as in the statement, choose $R > 0$ such that $K \subset B(e, R)$. Suppose, by contradiction, that $K \cap Kt \neq \emptyset$ for all $t \in L$, and choose $k_t \in K$ such that $k_t t \in K$. Then for any $p \in Kt$ ($p = p't$ with $p' \in K$) we have

$$d(e, p) \leq d(e, k_t t) + d(k_t t, p) = d(e, k_t t) + d(k_t, p')$$

since, by invariance of d , we have $d(k_t t, p) = d(k_t, p')$. We continue, noting that

$$\dots \leq d(e, k_t t) + d(k_t, e) + d(e, p') \leq 3R,$$

thus $Kt \subset B(e, 3R)$ for all $t \in L$. It follows that $KL = \{kt : t \in L, k \in K\} \subset B(e, 3R)$, so that $L \subset KL$ is bounded and hence relatively compact, a contradiction. \square

4.2. Restrictions to the Grauert tubes of G . Denoting the real n -ball by B^n , the Grauert tubes of $X = G$ of the form M_ϵ from (1) embed in \tilde{M}_δ in the following way:

$$(7) \quad M_\epsilon = G \times B^n \hookrightarrow G \times \mathbb{T} = \tilde{M}_\delta.$$

More specifically, in the notation of Sect. 3, for every fixed $z_0 = x_0 + iy_0$, the map $M_{\delta-y_0^2} \ni p \rightarrow (x_0, y_0, p) \in \tilde{M}_\delta$ is an embedding of $M_{\delta-y_0^2}$ as a complex submanifold of \tilde{M}_δ with $\text{codim}_{\mathbb{C}} M_\delta = 1$. Thus the restriction to the copy of M_ϵ at z_0 , denoted $M_\epsilon^{z_0}$, is a holomorphic map

$$\mathcal{O}(\tilde{M}_\delta) \ni f \mapsto f(z_0, \cdot) \in \mathcal{O}(M_\epsilon^{z_0})$$

and so elements of $L^2\mathcal{O}(\tilde{M}_\delta)$ restrict to elements of $\mathcal{O}M_\epsilon^{z_0}$ as long as $y_0^2 + \epsilon \leq \delta$.

The holomorphic functions constructed in Thm. 3.11 are, by construction, smooth in the closure of \tilde{M}_δ except along the orbit of the basepoint of f . Thus the restrictions of these functions are smooth in the closures of the $M_\epsilon^{z_0}$.

In the decompositions (5), choose the measure Q on $\mathbb{T} = \tilde{M}_\delta/G$ as a tensor product $dq \otimes d\lambda$ for a suitable measure $d\lambda$ on the cylinder T and dq on the slices $\{z_0 = \text{const.}\}$.

We will denote by t_*h the function

$$t_*h : z \longmapsto h(zt)$$

which is in $L^2\mathcal{O}(M_\epsilon)$ by the G -invariance of the measure on M_ϵ .

Lemma 4.2. *Let $h \in L^2(M_\epsilon)$ with $\|h\|_{L^2} = 1$. Let $L \subset G$ be a sequence, chosen in such a way that no subsequence $(t_k)_k \subset L$ has compact closure. It follows that the complex vector space $\langle h \rangle_L$ generated by $\{t_*h : t \in L\}$ is infinite-dimensional over \mathbb{C} .*

Proof. Suppose, by contradiction, that $\{t_*h \mid t \in L\}$ is contained in a finite-dimensional space. Since $\|t_*h\|_{L^2} = 1$ for all $t \in L$, there exists a sequence $(t_k)_k \subset L$ such that $(t_{k*}h)_k$ is convergent. Denote by $h_0 \in L^2(M_\epsilon)$ the limit of this sequence and note that $\|h_0\|_{L^2} = 1$.

Let $K \subset G$ be a fixed, arbitrary compact set containing the identity. We will obtain a contradiction to the assertion that $\|h_0\|_{L^2} = 1$ by showing that $h_0|_{\pi^{-1}(K)} \equiv 0$, where π is the G -equivariant projection map $\pi : M_\epsilon \rightarrow G$, cf. Sect. 2.2.

Fix $\epsilon_0 > 0$, and let $K_0 \supset K$ be a compact set such that the function h from the statement satisfies

$$\int_{M_\epsilon \setminus \pi^{-1}(K_0)} |h|^2 dt \otimes dq < \epsilon_0.$$

By a repeated application of Lemma 4.1, we obtain a sequence $(t_k)_k$ in L such that $K_0 \cap K_0 t_k = \emptyset$, thus

$$\pi^{-1}(K_0) \cap [\pi^{-1}(K_0)t_k] = \pi^{-1}(K_0 \cap K_0 t_k) = \emptyset$$

for all k , by the equivariance of π . It follows that

$$\int_{\pi^{-1}(K_0)} |t_{k*}h|^2 = \int_{\pi^{-1}(K_0)t_k} |h|^2 \leq \int_{M_\epsilon \setminus \pi^{-1}(K_0)} |h|^2 < \epsilon_0.$$

Since $K \subset K_0$ we have $\int_{\pi^{-1}(K)} |t_{k*}h|^2 < \epsilon_0$ and so $\|h_0\|_{L^2(\pi^{-1}(K))} \leq \epsilon_0$ since h_0 is the limit of $(t_{k*}h)_k$. By letting $\epsilon_0 \rightarrow 0$ we obtain $h_0|_{\pi^{-1}(K)} \equiv 0$. \square

Proposition 4.3. *Let M_ϵ be the Grauert tube of G . It follows that the space $L^2\mathcal{O}(M_\epsilon)$ is infinite-dimensional as a vector space over \mathbb{C} .*

Proof. Let $f \in L^2\mathcal{O}(\tilde{M}_\delta)$ be a non-trivial function provided by Theorem 3.11. From Fubini's theorem follows

$$\infty > \int_{\tilde{M}_\delta} |f(z_0, z)|^2 dt \otimes dQ = \int_{\tilde{M}_\delta} |f(z_0, z)|^2 dt \otimes dq \otimes d\lambda =$$

$$= \int_{\{y_0^2 < \delta\}} d\lambda(z_0) \int_{M_\epsilon^{z_0}} |f(z_0, z)|^2 (dt \otimes dq)(z)$$

where at each height y_0 we have chosen the radius $\epsilon = \epsilon(y_0) = \delta - y_0^2$. Thus the integral of $|f(c, z)|^2$ over $\tilde{M}_\epsilon \cap \{z_0 = c\} \cong M_{\epsilon - (\Im z_0)^2}$ is finite for almost every $c \in \{(\Im z_0)^2 < \epsilon\}$; in other words, for $\epsilon' < \epsilon$ in a set of positive measure, we obtain that $L^2\mathcal{O}(M_{\epsilon'}) \neq \{0\}$.

Let, then, h be a non-trivial element of $L^2\mathcal{O}(M_\epsilon)$. Since G is non-compact, we can choose a sequence $L \subset G$ as in Lemma 4.2 (for example, we can fix an exhaustion K_j of G by compact subsets and choose $t_j \in K_j \setminus K_{j-1}$). Then $\langle h \rangle_L$ is still contained in $L^2\mathcal{O}(M_\epsilon)$; by Lemma 4.2 follows that $\dim_{\mathbb{C}} L^2\mathcal{O}(M_\epsilon) = \infty$. \square

In the subspace of the Bergman space that we have constructed, we have a faithful unitary representation of G “modulo compact subgroups.” That is the content of the following

Theorem 4.4. *Let G be a unimodular group. Then, there is a representation \mathcal{R} of G in $L^2\mathcal{O}(M_\epsilon)$, where M_ϵ is a neighborhood of G in its complexification, such that $\ker \mathcal{R}$ is a compact subgroup of G . In particular, if G does not have compact subgroups then \mathcal{R} is a faithful representation.*

Proof. Define \mathcal{R} to be the natural representation of G in the space of unitary operators of $L^2\mathcal{O}(M_\epsilon)$, with M_ϵ as in the previous section, induced by $\mathcal{R}(t)f(z) = t_*f(z) = f(zt)$ for any $t \in G$ and $f \in L^2\mathcal{O}(M_\epsilon)$.

Let H be the kernel of \mathcal{R} , and choose a function $f \in L^2\mathcal{O}(M_\epsilon)$, $f \neq 0$; then $t_*f = f$ for all $t \in H$. In particular, the space generated by the translates of f by elements of H is 1-dimensional. By the same arguments as in Cor. 4.3, H must be relatively compact because otherwise, we could find a discrete subset $L \subset H$ with the property that every infinite subsequence of L is not relatively compact. \square

Remark 4.5. The representations constructed are strongly continuous.

5. THE THICKENED HEISENBERG GROUP $\mathbb{H}_3(\mathbb{R}) \hookrightarrow \mathbb{H}_3(\mathbb{C})$

We will describe here a simple, detailed example of the manifolds in question. For $\mathbb{K} = \mathbb{Z}$, \mathbb{R} , or \mathbb{C} , define

$$(8) \quad \mathbb{H}_3(\mathbb{K}) = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mid z_k \in \mathbb{K} \right\} \quad \text{and} \quad \mathfrak{h}_3(\mathbb{R}) = \left\{ \begin{pmatrix} 0 & \theta_1 & \theta_3 \\ 0 & 0 & \theta_2 \\ 0 & 0 & 0 \end{pmatrix} \mid \theta_k \in \mathbb{R} \right\}.$$

For $\mathbb{H}_3(\mathbb{R}) \hookrightarrow \mathbb{H}_3(\mathbb{C})$, close to the origin of $\mathbb{H}_3(\mathbb{C})$, the *slice through e* , $S_e \subset \mathbb{H}_3(\mathbb{C})$, is the exponential of the normal vectors of the identity in

the natural inclusion:

$$(9) \quad T_e \mathbb{H}_3(\mathbb{R}) \oplus \mathbb{H}_3(\mathbb{R})^\perp \hookrightarrow T_e \mathbb{H}_3(\mathbb{C}); \quad S_e = \exp T_e \mathbb{H}_3(\mathbb{R})^\perp \subset \mathbb{H}_3(\mathbb{C}).$$

I.e. $S \subset \mathbb{H}_3(\mathbb{C})$ consists of matrices of the form $\exp[i\Theta]$ with $\Theta = (\theta_{jk})_{jk}$ in $\mathfrak{h}_3(\mathbb{R})$. In [HHK], an invariant strongly plurisubharmonic function φ is constructed abstractly; here, the goal is to describe φ .

Expressing an arbitrary element Z in the form $Z = \exp[i\Theta]t$, with $t \in \mathbb{H}_3(\mathbb{R})$ and $\Theta \in \mathfrak{h}_3(\mathbb{R})$, as in the description (9), the definition of φ given in [HHK] becomes

$$(10) \quad \varphi(\exp[i\Theta]t) = \sum_{jk} \theta_{jk}^2.$$

Note that this definition makes the right- $\mathbb{H}_3(\mathbb{R})$ invariance of φ manifest. So, given $Z \in \mathbb{H}_3(\mathbb{C})$, consider the factorization

$$(11) \quad Z = \exp[i\Theta]t$$

with $t \in \mathbb{H}_3(\mathbb{R})$ and Θ a real matrix. Defining $X = \Re Z$ and $Y = \Im Z$, the identity $\exp[i\Theta] = \cos[\Theta] + i \sin[\Theta]$, provides

$$Z = \Re Z + i \Im Z = X + iY = (\cos[\Theta] + i \sin[\Theta])t.$$

Equating real and imaginary parts,

$$X = \cos[\Theta]t, \quad Y = \sin[\Theta]t,$$

thus

$$t = [\cos \Theta]^{-1}X = [\sin \Theta]^{-1}Y$$

and so

$$(12) \quad \Theta = \tan^{-1}(YX^{-1}) = YX^{-1} - \frac{(YX^{-1})^3}{3} + \frac{(YX^{-1})^5}{5} - \dots$$

We easily compute $\exp[i\Theta]$ for $\Theta \in \mathfrak{h}_3(\mathbb{R})$;

$$\exp \left[i \begin{pmatrix} 0 & \theta_1 & \theta_3 \\ 0 & 0 & \theta_2 \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & i\theta_1 & i\theta_3 - \theta_1\theta_2/2 \\ 0 & 1 & i\theta_2 \\ 0 & 0 & 1 \end{pmatrix},$$

thus the factorization (11) is explicitly

$$(13) \quad \begin{aligned} Zt = \exp[i\Theta]t &= Xt + iYt \\ &= \begin{pmatrix} 1 & i\theta_1 & i\theta_3 - (1/2)\theta_1\theta_2 \\ 0 & 1 & i\theta_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_1 & t_3 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

and any element of $\mathbb{H}_3(\mathbb{C})$ has a unique decomposition of this form. For a general $Z \in \mathbb{H}_3(\mathbb{C})$, computing YX^{-1} , which we need in (12), we get

$$(14) \quad YX^{-1} = \begin{pmatrix} 0 & y_1 & y_3 - x_2y_1 \\ 0 & 0 & y_2 \\ 0 & 0 & 0 \end{pmatrix},$$

where $z_k = x_k + iy_k$. The next step is to compute the arctangent, which is again easy because of the nilpotence. We obtain

$$\Theta = \tan^{-1}(YX^{-1}) = YX^{-1}.$$

Denoting by Θ^T the transpose of Θ , the sum of the squares of the elements of Θ as in Eq. (10) can be written $\text{tr}(\Theta^T \Theta)$ in general, but for our purposes, it is enough to read off

$$\begin{aligned} \varphi(Z) &= \text{tr}(\tan^{-1}(YX^{-1})^T \tan^{-1}(YX^{-1})) = \text{tr}((YX^{-1})^T YX^{-1}) \\ (15) \quad &= (\Im z_1)^2 + (\Im z_2)^2 + (\Im z_3 - \Re z_2 \Im z_1)^2, \end{aligned}$$

from (14). This is an invariant, strongly plurisubharmonic function in a neighborhood of $\mathbb{H}_3(\mathbb{R})$ in $\mathbb{H}_3(\mathbb{C})$. Now, as in Sect. 3, consider $(S^1 \times \mathbb{H}_3(\mathbb{R}))^{\mathbb{C}} \cong \mathbb{C}/\mathbb{Z} \times \mathbb{H}_3(\mathbb{C})$ with $\mathbb{H}_3(\mathbb{R})$ acting from the right, and trivially on the first factor:

$$\mathbb{H}_3(\mathbb{R}) \ni t : (z_0; Z) \longmapsto (z_0, Zt).$$

Defining a new function by $\tilde{\varphi}(z_0, Z) = (\Im z_0)^2 + \varphi(Z)$, an easy calculation shows that $\tilde{M}_\epsilon = \{\tilde{\varphi} < \epsilon\}$ is strongly pseudoconvex as long as $\epsilon < 1$. With the definition (3), at $z^0 = (z_0, z_1, z_2, z_3) = (i\epsilon/2, 0, 0, 0) \in \{\tilde{\varphi} = \epsilon^2/4\}$, $\tilde{\varphi}$ induces a Levi polynomial proportional to

$$f(z) := 4\epsilon - 1 + 4iz_0(2\epsilon - 1) + 4z_0^2 + 4z_1^2 + 4z_2^2 + 4z_3^2,$$

as in Lemma 3.1. The action of $t \in \mathbb{H}_3(\mathbb{R})$ on f is given by

$$f(z_t) = 4\epsilon - 1 + 4iz_0(2\epsilon - 1) + 4z_0^2 + 4(z_1 + t_1)^2 + 4(z_2 + t_2)^2 + 4(z_3 + z_1 t_2 + t_3)^2$$

The convolution of χf^{-1} by a convolution kernel $\Delta \in C^\infty(\mathbb{H}_3(\mathbb{R}))$ is approximated in a neighborhood of the basepoint z^0 by

$$(R_\Delta \chi f^{-1})(z) \sim \int dt [4\epsilon - 1 + 4iz_0(2\epsilon - 1) + 4z_0^2 + 4(z_1 + t_1)^2 + 4(z_2 + t_2)^2 + 4(z_3 + z_1 t_2 + t_3)^2]^{-1}$$

where we have shortened $dt_1 dt_2 dt_3$ to simply dt and with all the integrals over small intervals $t_k \in (-\epsilon_k, \epsilon_k)$, $k = 1, 2, 3$. Now consider the path from inside the manifold to the basepoint $(i\epsilon/2, 0, 0, 0)$ given by $s \mapsto (is/2, 0, 0, 0)$, $s \rightarrow \epsilon^-$. Along this path, the convolution simplifies to

$$\begin{aligned} (R_\Delta f^{-1})(i\epsilon/2, 0, 0, 0) &\sim \int dt [4\epsilon - 1 - 2(2\epsilon - 1)s - s^2 + 4t_1^2 + 4t_2^2 + 4t_3^2]^{-1} \\ (16) \quad &\sim \int_0^\delta \frac{r^2 dr}{[\sigma + 4r^2]^\tau} \end{aligned}$$

with $\sigma \rightarrow 0$ as $s \rightarrow \epsilon^-$, as in Prop. 3.9.

Note that since $\mathbb{H}_3(\mathbb{R})$ has a cocompact discrete subgroup, $\mathbb{H}_3(\mathbb{Z})$, the construction of L^2 holomorphic functions in [GHS] yields other elements of $L^2\mathcal{O}(\{\varphi < \epsilon\})$, these having local peak points in the boundary and with no thickening. Of course, in such a concrete example, one could also construct elements of $L^2\mathcal{O}$ by hand. Also note that a generic unimodular Lie group possesses no discrete cocompact subgroups, [M].

We end this example with a question: Is it possible to follow through with the restriction in Sect. 4.2 to determine the representation actually obtained here and then relate that to the representations of $\mathbb{H}_3(\mathbb{R})$ in [T]?

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